# Prediction of Transport Processes within Porous Media: Creeping Flow Relative to a Fixed Swarm of Spherical Particles

A geometric model for an homogeneous swarm of spherical particles was introduced by us in an earlier paper and successfully employed in a theoretical investigation of diffusive transport processes occurring therein.

The same geometric model is used here in a theoretical study of the hydrodynamic transport process occurring within a fixed swarm of spherical particles. The product of the application of this model to the problem of incompressible, creeping fluid flow within an homogeneous swarm of impermeable spherical particles may be regarded as a logical extension of the well-known Brinkman model; it permits physical representation and rigorous mathematical solution, yielding predictions which are in good agreement with experimental data throughout the entire porosity range. For porosities in excess of 0.7, the predictions agree closely with those obtained by means of Happel's free surface model.

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## SCOPE

All fundamental transport processes occurring in the interstices of porous media may be classified into two categories: The diffusive flow processes (molecular or ionic diffusion, electric conduction, and thermal conduction, and the hydrodynamic flow process (momentum transport). Compound transport phenomena encountered within porous media simultaneously involve two or more of the fundamental transport processes.

A factual, immediate theoretical investigation of these phenomena within even the simplest homogeneous and isotropic porous medium (a randomly arranged swarm of spherical particles) is precluded by the immensely complicated, usually undefined, geometric structure of the porous medium. Hence, a geometric model for the porous medium must be introduced. The usefulness of such a geometric model is in the restrained simplification of the complicated geometric structure, thus permitting a (necessarily simplified) physical representation which provides insight into the essential features of the transport process and which, upon successful application of mathematical means, yields predictions for relationships between practically important (and physically measurable) geometric parameters of the porous medium and physical parameters of the transport process under investigation.

Numerous geometric models have been proposed and presented in the literature, each to study some specific transport process. No attempt is made here to systematically classify the geometric models of porous media according to basic type. Two fundamental types are mentioned, however, to emphasize one further specialization of these models, namely, the intended, or useful, range of porosity: The famous Kozeny model concentrates upon the totally enclosed pore space and replaces it by a bundle of capillaries with circular cross sections; it and all its modifications clearly are intended for relatively low values of porosity. The model by Maxwell (1892) which considers but one typical grain in unbounded space is typical for another basic type of geometric model; predictions obtained by its application are derived for porosity $\epsilon = 1$, being most reliable in some neighbourhood of this value.

Relatively crude geometric models of porous media oversimplify the geometry and thus lead to simple physical representations which can then be solved by relatively modest mathematical efforts; however, the physical representation is unrealistic, any physical insight into the transport process is questionable, and the correlations derived are of limited benefit. Obviously, a compromise must be sought. The best geometric model is that which imposes a minimum of geometric simplifications yet leads to a physical representation which affords a mathematical solution.

Most geometric models developed and proposed for the theoretical study of one category of fundamental transport processes are unproductive when used with theoretical investigations of the other category. This observation is particularly disappointing in view of the need for theoretical studies of the many practically important compound transport processes occurring within porous media, which simultaneously involve both diffusive and hydrodynamic transport processes and which require one geometric model which can serve both categories.

In an earlier paper (Neale and Nader, 1973), the authors proposed a geometric model for an homogeneous swarm of spherical particles and demonstrated the usefulness of the model in the theoretical study of diffusive transport processes occurring in the interstices of the swarm for the entire porosity range. In this investigation, the same model is used in a theoretical study of the hydrodynamic flow process occurring within an homo-

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geneous swarm of fixed spherical particles, for incompressible Newtonian fluids in the creeping flow regime (low Reynolds numbers), and for the entire porosity range.

A series of carefully executed permeability measurements using spherical particles of the highest available uniformity with respect to shape, size, and surface characteristics (namely, precision ground, 1 mm diameter, stainless steel bearing balls) is described in this work.

## CONCLUSIONS AND SIGNIFICANCE

It was the original objective of our theoretical study of transport processes, occurring within unconsolidated homogeneous and isotropic porous media composed of spherical particles, to develop one geometric model which would serve well for either category of transport processes (the diffusive flow processes and the hydrodynamic flow process) over the entire practically encountered porosity range. The geometric model has been introduced in an earlier paper (Neale and Nader, 1973) and successfully employed in the theoretical study of diffusive transport processes.

In this study, it is shown that the same geometric model is also productive in the theoretical study of the hydrodynamic flow process relative to a fixed swarm of monosized spherical particles. The geometric model effects a rather limited geometric simplification of the immensely complicated homogeneous and isotropic porous medium. Thus, the physical representation of the problem is not unrealistically simplified. As a consequence, the mathematical representation is not of a simple nature. However, a mathematical solution could be obtained. This mathematical solution is being used to develop a prediction for the permeability of the porous medium and to find the flow field near the reference sphere. Using the solution, a fully predictive relationship between the permeability $k$, the porosity $\epsilon$, and the particle radius $R$ could be developed. The good agreement of the predictions with experimental data (see Figure 4) lends confidence to the entire process of geometric modeling and physical representation of the system. The experimental data obtained in this work (see Table 2) is consistent with that appearing elsewhere in the literature.

By now, the usefulness of the geometric model has been tested for each category of fundamental transport processes for the entire porosity range. Thus, this geometric model is best qualified for future theoretical investigations of important compound transport processes (occurring within an homogeneous and isotropic swarm of spherical particles) which involve two or more fundamental transport processes of both categories. For such investigations, the microscopic flow field near the reference sphere needs be known. Figure 6 shows the flow field near the reference sphere as predicted by the solution of the hydrodynamic problem.

In the description of the hydrodynamic flow problem, the authors employ the Brinkman equation [Equation (5)] instead of the widely used Darcy equation [Equation (4)] to describe the macroscopic flow field throughout the porous medium. This choice is considered necessary to meet the conditions in the porous medium at a boundary next to an open fluid; it is well supported by the reported results (see Figure 4). More recent, as yet unpublished, work regarding boundary conditions for porous media in general further supports the above choice and explains the situations.

The theory developed here for predicting fluid flow through a fixed bed of spherical particles is also valid for predicting fluidization of a fixed swarm of spheres and hindered settling of a fixed suspension of spheres since all three processes involve the relative motion of a continuous fluid phase and a fixed particulate phase. However, very seldom during fluidization or sedimentation experiments would the solid particles remain fixed relative to one another. Generally speaking, they would rotate and translate within the swarm as a result of hydrodynamic interactions between the particles. This introduces mathematical complications which are not of a trivial nature. Fortunately, these complications do not arise in the present study, which is concerned exclusively with fixed swarms.

## THE FUNDAMENTAL CONCEPT OF THE GEOMETRIC MODEL AND ITS APPLICATION

An unbounded, homogeneous, and isotropic porous medium of porosity $\epsilon$, composed of spherical particles, is depicted by a typical cross section shown in Figure 1. Any sphere is arbitrarily selected as the reference sphere, with its associated pore space. According to the geometric modeling procedure, we locally rearrange the porous medium as indicated in Figure 2, a cross section through the reference sphere. The reference sphere of radius $R$ is surrounded by a concentric spherical shell comprising the associated pore space, having outer radius $S$. The reference sphere together with its spherical shell of associated pore space will be referred to as the unit cell; it is embedded within the exterior porous mass of porosity $\epsilon$.

In order to preclude any local macroscopic disturbance of the uniform porosity $\epsilon$ of the macroscopically homogeneous porous medium, we must ensure that the unit cell contains exactly the pore space properly associated with the volume of the reference sphere; in short, the porosity of the unit cell must be equal to that of the porous medium. This necessitates that

$$S/R = (1 - \epsilon)^{-1/3} \qquad (1)$$

An inherent advantage of the geometric model lies in the fact that the macroscopically homogeneous and isotropic characteristics of the porous medium are preserved in the modeled system. Observe that the geometric model differs fundamentally from the two types of geometric model mentioned earlier: This model employs a modeled pore space, a typical grain, and the porous medium.

The geometrically modeled system provides the basis for a theory of any transport process occurring within the homogeneous and isotropic porous medium. In essence, the technique of solution is to solve the requisite partial and the continuity equations. Thus, for an incompressible Newtonian fluid and creeping flow conditions:

$$\mu \nabla^2 \underline{u} = \nabla p \quad R < r < S \tag{2}$$

$$\nabla \cdot \underline{u} = 0 \quad R < r < S \tag{3}$$

Appropriate boundary conditions must be specified for $r \to R$ and for $r \to S$ to guarantee a unique solution within the spherical shell.

Within the exterior porous mass (henceforth treated as a continuum), a macroscopic description of the flow field is required. A macroscopic equation which describes incompressible, creeping flow of a Newtonian fluid of viscosity $\mu$ through a macroscopically homogenous and isotropic porous medium of permeability $k$ is the well-known Darcy equation:

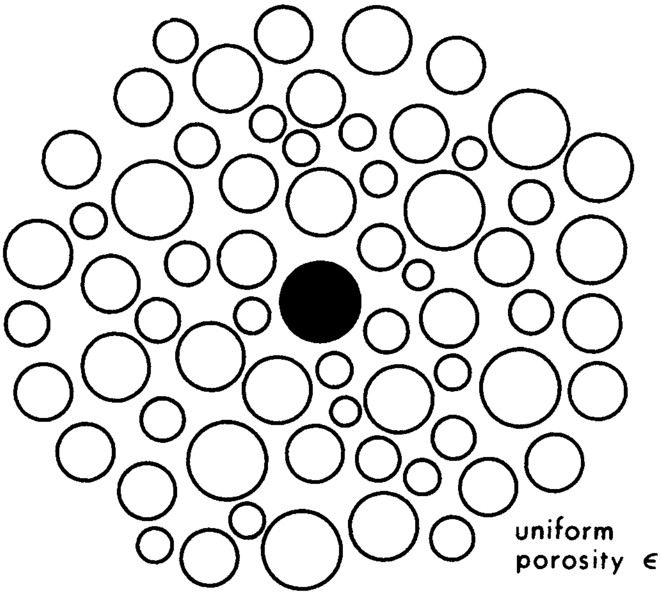
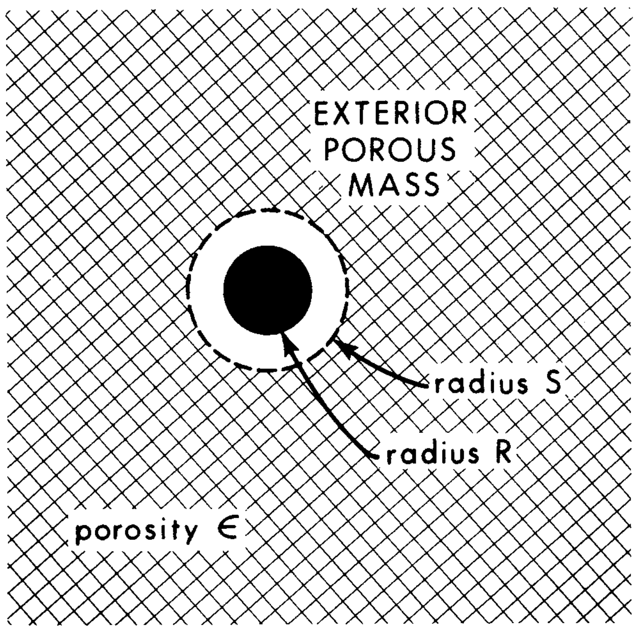



**Fig. 1. An homogeneous and isotropic swarm of spheres (cross section through center of reference sphere).**




$$\frac{S}{R} = (1-\epsilon)^{-1/3}$$

**Fig. 2. The proposed model for an homogeneous and isotropic swarm of spheres (cross section through center of reference sphere).**

differential equations both within the pore space of the unit cell and within the exterior porous mass* and to connect the respective solutions by means of physically realistic boundary conditions at the separating interface and by uniformity conditions.

The presented application of the geometric model to the hydrodynamic flow problem may be regarded as a logical generalization of the widely quoted Brinkman model (Brinkman, 1947), which envisages a reference sphere embedded directly within the homogeneous porous mass without any provision for the associated pore space. The insertion of the spherical shell of associated pore space is of fundamental importance.* It not only enables us to maintain macroscopic homogeneity during the modeling procedure but more important it also provides the all-important model pore space in which the particular transport process can be studied, and the derived solution be connected with the solution in the exterior porous mass. The provision of the spherical shell of associated pore space results in a more realistic model, causes a more difficult mathematical representation, and alleviates the anomalous characteristics of the Brinkman solution.

## MATHEMATICAL DESCRIPTION OF THE MODELED PHYSICAL SYSTEM

We have now, for the theoretical study of incompressible, creeping fluid flow, instead of the homogeneous and isotropic porous medium composed of impermeable spherical particles (see Figure 1), the geometrically modeled system shown in Figure 2. In order to study the behavior of the system we choose to impose a uniform pressure gradient (in the $+x$-direction) as shown in Figure 3, thus introducing a uniform, axisymmetric macroscopic flow field $\underline{\hat{U}}$, defined by $\underline{\hat{U}} = [\hat{U}, 0, 0]$ in Cartesian coordinates $[x, y, z]$.

### Fundamental Partial Differential Equations

Within the spherical shell of associated pore space, the prevailing flow field is governed by the Navier-Stokes

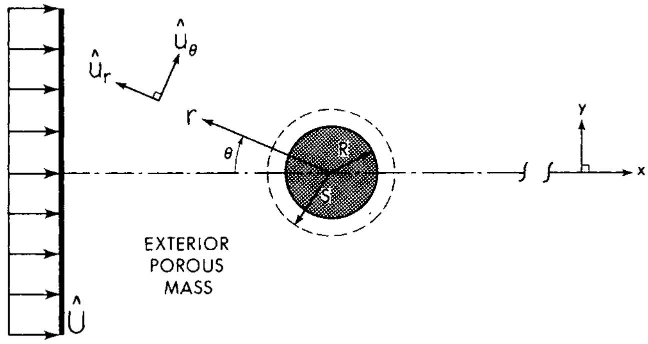



**Fig. 3. Description of the modeled system for fluid flow through an homogeneous swarm of spheres (cross section through center of reference sphere).**

* In a later paper, Brinkman (1948) made an unsuccessful attempt to incorporate a concentric shell of pore space possessing empirically determined dimensions.

* Occasionally, the reference sphere itself must be considered, for instance, when dealing with conductive spherical particles (electric conduction, heat conduction) or with permeable spherical particles (diffusion, fluid flow).

$$-\frac{\mu}{k}\hat{\underline{u}} = \nabla \bar{p} \quad S < r < \infty \tag{4}$$

where $\wedge$ denotes an assembly-averaged quantity within the porous medium, and — denotes an interstitially-averaged quantity within the porous medium.

In hydrodynamics, one generally requires continuity of the pressure, of the flow vector, and of the shear tensor throughout the fundamental domain of the problem—in particular (in the limit) when approaching boundary surfaces which separate subdomains of the problem. When applying this principle to our problem, one would require at the boundary sphere ($r = S$) which separates the unit cell from the exterior porous mass: Continuity of (a) the pressure, (b) the component of $\underline{u}$ orthogonal to the sphere, (c) the other component of $\underline{u}$, and (d) the shear component tangential to the sphere. Considerations of this type expose a deficiency of the Darcy equation (4). Actually, there is mounting evidence, both experimental and theoretical, which suggests that the Darcy equation (4) will sometimes provide an unsatisfactory description of the hydrodynamic conditions, particularly near boundaries of a porous medium. Beavers et al. (1967, 1970) have experimentally demonstrated the existence of shear within the porous medium near a surface where the porous medium is exposed to a freely flowing fluid, thus forming a zone of shear-influenced fluid flow. However, the Darcy equation (4) cannot predict the existence of such a boundary zone, as no macroscopic shear term is included in this equation (Joseph and Tao, 1964). Recently, Slattery (1969) and Tam (1969) have demonstrated

$$-\frac{\mu}{k}\hat{\underline{u}} + \mu\nabla^2\hat{\underline{u}} = \nabla \bar{p} \quad S < r < \infty \tag{5}$$

and

$$\nabla \cdot \hat{\underline{u}} = 0 \quad S < r < \infty \tag{6}$$

as the governing equations for incompressible, creeping flow of a Newtonian fluid within an isotropic and homogeneous porous medium. More recently, Saffman (1971), Lundgren (1972), and Childress (1972) have presented elaborate statistical justification of Equations (5) and (6).

Equation (5) was originally proposed by Brinkman (1947, 1949); hence, we shall refer to it as the Brinkman equation. It is physically consistent with the above mentioned, experimentally-observed boundary shear zone on account of its shear term $\mu\nabla^2\hat{u}$, and it is mathematically compatible with the Navier-Stokes equation (2) at the surface of the unit cell ($r = S$). Also, Equation (5) does properly become Equation (2) as the porosity tends towards 1 and the permeability grows without bound, a characteristic which Equation (4) lacks.

Probably the most important argument in favor of the Darcy equation (4), and against the Brinkman equation (5), emphasizes the fact that (5) is an immensely more difficult equation to solve. However, it is not difficult to demonstrate firstly that for porous media of relatively low permeability (low porosity) the first term of (5) dominates the second term, and secondly that outside a zone next to the boundaries of the porous medium the contribution of the term $\mu\nabla^2\hat{\underline{u}}$ becomes insignificant. The thickness of the boundary zone is actually quite small, of the order of $\sqrt{k}$ (Saffman, 1971); hence, the Brinkman equation (5) effectively reduces to the Darcy equation (4) within the biggest portion of the porous medium outside the above mentioned thin boundary zones.

As indicated earlier, the effectiveness of the physical representation based upon the geometric model of the porous medium depends very much upon the precise formulation of physically realistic boundary conditions at the surface of the unit cell ($r = S$) to connect the respective solutions in the spherical shell of associated pore space and in the exterior porous mass. It is for this reason that in our investigation we cannot accept the approximation (4) of the Brinkman equation (5). The postulation of (2) and (4) in the respective regions requires the omission of one of the boundary conditions, leads to a mathematically simpler problem, and yields unsatisfactory results (see below). Consequently, we shall postulate (2) and (3) within the spherical shell, and (5) and (6) within the exterior porous mass.

#### Stipulated Boundary Conditions

Any disturbance introduced by the locally restricted modeling procedure must have localized effects. In particular, this implies that

$$\lim_{r\to\infty} \hat{u}_r(r, \Theta, \varphi) = -\hat{U}\cdot\cos\Theta \quad 0 \leq \Theta < 2\pi, \quad -\pi \leq \varphi < \pi \tag{7}$$

$$\lim_{r\to\infty} \hat{u}_\Theta(r, \Theta, \varphi) = \hat{U}\cdot\sin\Theta \quad 0 \leq \Theta < 2\pi, \quad -\pi \leq \varphi < \pi \tag{8}$$

This investigation is restricted to solid (impermeable) spherical particles. This necessitates that

$$u_r(R^+, \Theta, \varphi) = 0 \quad 0 \leq \Theta < 2\pi, \quad -\pi \leq \varphi < \pi \tag{9}$$

$$u_\Theta(R^+, \Theta, \varphi) = 0 \quad 0 \leq \Theta < 2\pi, \quad -\pi \leq \varphi < \pi \tag{10}$$

Considerations of continuity at the interface between the spherical shell of associated pore space and the exterior porous mass require that

$$u_r(S^-, \Theta, \varphi) = \hat{u}_r(S^+, \Theta, \varphi) \quad 0 \leq \Theta < 2\pi, \quad -\pi \leq \varphi < \pi \tag{11}$$

$$p(S^-, \Theta, \varphi) = \bar{p}(S^+, \Theta, \varphi) \quad 0 \leq \Theta < 2\pi, \quad -\pi \leq \varphi < \pi \tag{12}$$

Acknowledging a viscous shear effect within a boundary zone in the exterior porous mass, as indicated by (5), and discussed above, requires that

$$u_\Theta(S^-, \Theta, \varphi) = \hat{u}_\Theta(S^+, \Theta, \varphi) \quad 0 \leq \Theta < 2\pi, \quad -\pi \leq \varphi < \pi \tag{13}$$

$$\tau_{r\Theta}(S^-, \Theta, \varphi) = \hat{\tau}_{r\Theta}(S^+, \Theta, \varphi) \quad 0 \leq \Theta < 2\pi, \quad -\pi \leq \varphi < \pi \tag{14}$$

Boundary conditions of this nature were first postulated by Brinkman (1948) and have been used more recently by Ooms et al. (1970), by Spielman and Goren (1968), and by Neale et al. (1973). The proper formulation of the conditions at the boundaries of porous media received theoretical attention from Tam (1969), Lundgren (1972), and Saffman (1971).

## THE MATHEMATICAL SOLUTION

The mathematical formulation of the modeled system is specified by (2) and (3) within the spherical shell of associated pore space ($R < r < S$), and by (5) and (6) within the unbounded, exterior porous mass ($S < r < \infty$), together with the boundary conditions (7) to (14). The derivation of the solution, $\underline{u}(r, \Theta, \varphi)$ in $R < r < S$ and $\hat{\underline{u}}(r, \Theta, \varphi)$ in $S < r < \infty$, of this rather difficult boundary value problem is summarized in Appendix A.*

The flow field $\underline{u}(r, \Theta, \varphi)$ in the spherical shell of associated pore space can be used to obtain a mathematical expression for the hydrodynamic force $F_{sph}$ acting upon the reference sphere:

$$F_{sph} = 6\pi\mu\hat{U}R \cdot \xi(\alpha, \beta) \tag{15}$$

where

$$\xi(\alpha,\beta) = \frac{4(-6\beta^6 - 21\beta^5 - 45\beta^4 - 45\beta^3 + 5\beta^4\alpha^2 + 5\beta^3\alpha^2 + \beta\alpha^5 + \alpha^5)}{(-4\beta^6 - 24\beta^5 - 180\beta^4 - 180\beta^3 + 9\beta^5\alpha + 45\beta^4\alpha - 10\beta^3\alpha^3 + 180\beta^3\alpha - 30\beta^2\alpha^3 + 9\beta\alpha^5 - 4\alpha^6 + 9\alpha^5)} \tag{16}$$

designates the influence function in terms of the dimensionless quantities

$$\alpha = R/\sqrt{k} \tag{17}$$

$$\beta = S/\sqrt{k} \tag{18}$$

By means of the relationship (1), we can express $\beta$ in terms of $\alpha$ and $\epsilon$

$$\beta = \alpha \cdot (1 - \epsilon)^{-1/3} \tag{19}$$

We conclude that $\xi$ is, actually, a function of $R/\sqrt{k}$ and $\epsilon$

$$\xi = \xi\left(\frac{R}{\sqrt{k}}, \frac{R}{(1-\epsilon)^{1/3}\sqrt{k}}\right) \tag{20}$$

One can mathematically demonstrate that $\xi(\alpha, \beta) \rightarrow 1$ when $\epsilon \rightarrow 1$ or, equivalently, when $S/R \rightarrow \infty$. This shows that $F_{sph}$ properly approaches the hydrodynamic force predicted by Stokes law for incompressible, creeping fluid flow past a single solid spherical particle of radius $R$:

$$F_{sph} \rightarrow F_{Stokes} = 6\pi\mu\hat{U}R \quad \text{as} \quad S/R \rightarrow \infty \tag{21}$$

The influence function $\xi$ provides a measure of the deviation from Stokes law for a solid spherical particle of radius $R$ within an homogeneous and isotropic swarm of solid spherical particles having porosity $\epsilon$ and permeability $k$.

## PERMEABILITY OF AN HOMOGENEOUS AND ISOTROPIC SWARM OF MONOSIZED SPHERICAL PARTICLES

A very important hydraulic parameter of an homogeneous and isotropic swarm of spherical particles is its permeability. We shall use the result (15) with (16) to derive a theoretical prediction of the permeability of an homogeneous and isotropic swarm of monosized solid spherical particles of radius $R$, having porosity $\epsilon$.

We subject the unbounded, homogeneous, and isotropic swarm of spherical particles to the uniform pressure gradient (in the $+x$-direction)

$$\nabla \bar{p}(x,y,z) = \frac{\bar{p}(x+\Delta x, y, z) - \bar{p}(x,y,z)}{\Delta x} = -\frac{\Delta \bar{p}}{\Delta x} \tag{22}$$

with $\Delta \bar{p} > 0$. The resulting velocity $\hat{U}$ will be uniformly constant throughout; hence, $\nabla^2 \hat{U} = 0$, the second term of the Brinkman equation (5) vanishes, and we find the relationship between $\hat{U}$ and $\Delta \bar{p}/\Delta x$

$$-\frac{\mu}{k}\hat{U} + 0 = \nabla \bar{p} = -\frac{\Delta \bar{p}}{\Delta x} \tag{23}$$

Next we choose a sufficiently large, orthogonal cylinder of thickness $\Delta x$ and invariant cross section of area $A$ which contains exactly $N$ spherical particles of radius $R$ and the properly associated pore space. Thus,

$$N\frac{4\pi R^3}{3} = (1-\epsilon)A\Delta x \tag{24}$$

The hydrodynamic force $F_{cyl}$ acting on the cylinder is given by

$$F_{cyl} = A \cdot \bar{p}(x,y,z) - A \cdot \bar{p}(x+\Delta x, y, z) = A \cdot \Delta \bar{p} \tag{25}$$

Substituting in (25) for $\Delta \bar{p}$ by (23), for $A \cdot \Delta x$ by (24), and for $k$ by (17), one obtains

$$F_{cyl} = 6\pi\mu\hat{U}RN\frac{2\alpha^2}{9(1-\epsilon)} \tag{26}$$

Alternatively, one may consider each of the $N$ spherical particles of radius $R$ in the cylinder as a reference sphere and determine the hydraulic force acting upon it by (15) with (16); for all $N$ spherical particles one determines thus the total hydraulic force acting upon the cylinder:

$$F_{total} = N \cdot F_{sph} = N \cdot 6\pi\mu\hat{U}R \cdot \xi(\alpha,\beta) \tag{27}$$

$F_{total}$, by (27) must be equal to $F_{cyl}$, by (26); hence,

$$2\alpha^2 - 9(1-\epsilon) \cdot \xi(\alpha,\beta) = 0 \tag{28}$$

Substitution for $\beta$ by (19) yields the implicit function

$$2\alpha^2 - 9(1-\epsilon) \cdot \xi\left(\alpha, \frac{\alpha}{(1-\epsilon)^{1/3}}\right) = 0 \tag{29}$$

which relates the dimensionless quantities $\alpha = R/\sqrt{k}$ and $\epsilon$. The sought prediction for the permeability $k$ of the homogeneous and isotropic swarm of monosized impermeable spherical particles of radius $R$, having porosity $\epsilon$, is implicitly contained in (29). Unfortunately, (29) with (16) is a very complicated function indeed; it cannot be reformed into the desirable explicit form, say

$$\alpha = \alpha(\epsilon) \quad 0 < \epsilon < 1 \tag{30}$$

Numerical techniques and a digital computer were engaged to extract the relationship $\alpha(\epsilon)$ from (29). The prediction $\alpha(\epsilon)$ is reported in Table 1, and shown in Figure 4. The function $\alpha(\epsilon)$ is single-valued; also, when $\epsilon \rightarrow 0$, $\alpha(\epsilon) \rightarrow \infty$, that is (for bounded $R$) $k \rightarrow 0$, and when $\epsilon \rightarrow 1$, $\alpha(\epsilon) \rightarrow 0$, that is (for $R > 0$) $k \rightarrow \infty$.

It is instructive to compare the hydraulic force acting upon the cylinder with the force one would calculate by the use of Stokes law for every spherical particle of radius $R$ in the cylinder (thus disregarding the mutual influences of the several spherical particles). The ratio of these forces is given by

$$W = \frac{N \cdot F_{Stokes}}{F_{total}} = \frac{N \cdot 6\pi\mu\hat{U}R}{N \cdot 6\pi\mu\hat{U}R \cdot \xi(\alpha,\beta)} = \frac{1}{\xi(\alpha,\beta)}$$

$$= \frac{9(1-\epsilon)}{2[\alpha(\epsilon)]^2} = W(\epsilon) \tag{31}$$

using the relationship $\alpha(\epsilon)$ predicted by (29). The relationship $W(\epsilon)$ predicted by (31) is reported in Table 1 and shown in Figure 5.

* Appendix A has been deposited as Document No. 02375 with the National Auxiliary Publications Service (NAPS), c/o Microfiche Publications, 305 E. 46 St., N. Y., N.Y. 10017 and may be obtained for \$1.50 for microfiche or \$5.00 for photocopies.

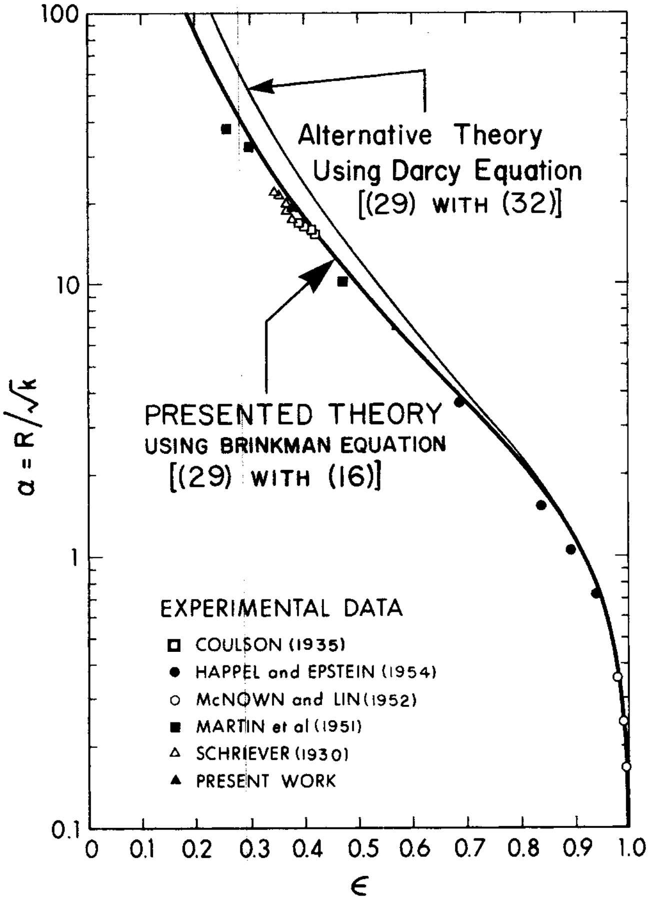



**Fig. 4. The predicted dependence of $\alpha$ on $\epsilon$ for monosized spheres: comparison with experimental data.**

## COMPARISON OF THE PREDICTIONS WITH EXPERIMENTAL RESULTS

Figure 4 shows the relationship $\alpha(\epsilon)$, as predicted by (29), together with representative experimental data points taken from the literature, and data obtained by experiments. The overall agreement between prediction and observation can be seen to be very satisfactory.

In keeping with the manner of presentation found in the literature, both predictions and experimental results are also presented in the alternative form of Figure 5 which displays the relationship $W(\epsilon)$ as predicted by Equations (29) and (31).

In connection with Figures 4 and 5, it should be mentioned that the data of Happel and Epstein (1954) and the data of Martin et al. (1951) were obtained using regular arrays of spherical particles.

## AN ALTERNATIVE PHYSICAL DESCRIPTION, USING THE DARCY EQUATION

The formulation of the physical representation used in the theoretical study is obtained in two steps: The geometric model transforms the undefined geometric structure of the porous medium into a well-defined geometric arrangement. Thus it becomes possible to employ properly selected fundamental physical laws and concepts to describe the transport process both in the unit cell and in the exterior porous mass. Both minimal geometric simplification by the geometric modeling process, and appropriate physical description, are necessary for the production of a good physical representation.

Thus, one might insist that the Darcy equation (4) ought to be used to describe the incompressible, creeping fluid flow in the exterior porous mass. This would require the abandoning of the boundary condition (14) to alleviate inconsistencies at the boundary which separates the unit cell from the exterior porous mass, as noted earlier. The resulting boundary value problem would be mathematically simpler. The solution (15) would hold; however, $\xi(\alpha, \beta)$ by (16) would have to be replaced by

Table 1. The Predicted Dependence of $\alpha$ and $W$ on $\epsilon$ for Homogeneous Swarms of Monosized Spheres

| $\epsilon$ | $\alpha(\epsilon)$ | $W(\epsilon) = 1/\xi(\alpha, \beta)$ |
|---|---|---|
| 1.0 | 0.0 | 1.0 |
| 0.9999999999 | $0.2003 \times 10^{-4}$ | 0.9994 |
| 0.99999999 | $0.2124 \times 10^{-3}$ | 0.9968 |
| 0.999999 | $0.2137 \times 10^{-2}$ | 0.9850 |
| 0.9999 | $0.2200 \times 10^{-1}$ | 0.9300 |
| 0.999 | $0.7282 \times 10^{-1}$ | 0.8487 |
| 0.99 | 0.2584 | 0.6738 |
| 0.95 | 0.7071 | 0.4501 |
| 0.90 | 1.185 | 0.3207 |
| 0.85 | 1.684 | 0.2379 |
| 0.80 | 2.247 | 0.1782 |
| 0.75 | 2.908 | 0.1331 |
| 0.70 | 3.708 | $0.9819 \times 10^{-1}$ |
| 0.65 | 4.706 | $0.7112 \times 10^{-1}$ |
| 0.60 | 5.986 | $0.5023 \times 10^{-1}$ |
| 0.55 | 7.677 | $0.3436 \times 10^{-1}$ |
| 0.50 | 9.974 | $0.2262 \times 10^{-1}$ |
| 0.45 | 13.19 | $0.1423 \times 10^{-1}$ |
| 0.40 | 17.83 | $0.8496 \times 10^{-2}$ |
| 0.35 | 24.74 | $0.4780 \times 10^{-2}$ |
| 0.30 | 35.44 | $0.2508 \times 10^{-2}$ |
| 0.25 | 52.93 | $0.1205 \times 10^{-2}$ |
| 0.20 | 83.87 | $0.5118 \times 10^{-3}$ |
| 0.15 | 146.0 | $0.1794 \times 10^{-3}$ |
| 0.10 | 302.7 | $0.4419 \times 10^{-4}$ |
| 0.05 | 967.9 | $0.4564 \times 10^{-5}$ |
| 0.0 | $\infty$ | 0.0 |

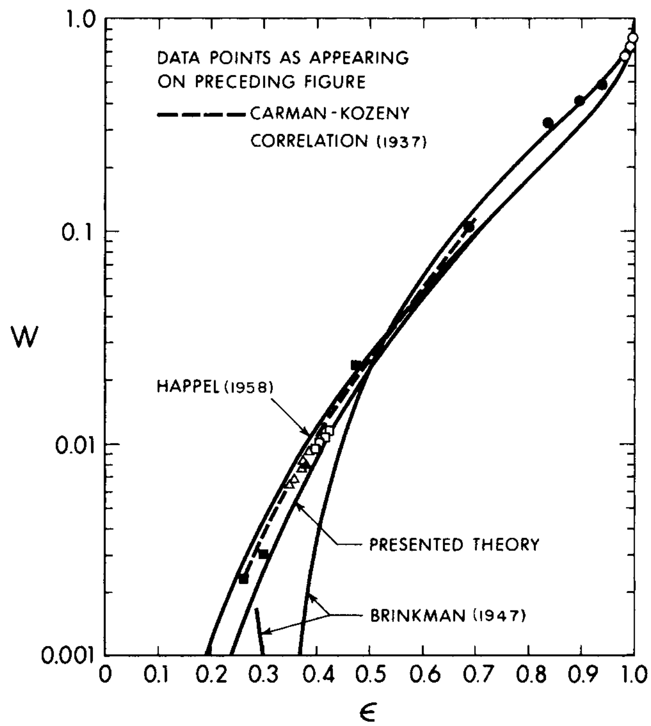



**Fig. 5. The dependence of w on $\epsilon$ for monosized spheres: comparison with previous work.**

$$\xi_{\text{Darcy}}(\alpha, \beta) = \frac{4(-6\beta^6 - 30\beta^4 + 5\beta^4\alpha^2 + \beta\alpha^5)}{(-4\beta^6 - 120\beta^4 + 9\beta^5\alpha - 10\beta^3\alpha^3 + 90\beta^3\alpha + 9\beta\alpha^5 + 30\beta\alpha^3 - 4\alpha^6)} \quad (32)$$

Using this result, one could, by (29), determine a prediction $\alpha_{\text{Darcy}}(\epsilon)$. The latter has been calculated and was judged unsatisfactory; it has been included in Figure 4 for comparison.

## THE MICROSCOPIC FLOW FIELD IN THE MODEL

The productivity and the reliability of the geometric model when used in the theoretical study of the hydrodynamic flow process occurring within an homogeneous and isotropic swarm of impermeable spherical particles has been demonstrated by the consistent and good prediction of the permeability for such systems over the entire porosity range. Possibly of greater importance than the prediction of the permeability for an homogeneous and isotropic swarm of spherical particles, and far more difficult to verify directly, is the prediction of the microscopic fluid flow pattern near the reference sphere; this information is needed for theoretical studies of compound transport processes occurring within an homogeneous and isotropic swarm of spherical particles when the hydraulic transport process is involved.

Streamlines calculated for a system having porosity $\epsilon = 0.7$ ($S/R = 1.494$, $\alpha = 3.708$ and $\beta = 5.540$), using the formulae presented in Appendix A, are displayed in Figure 6. The presented field of streamlines is typical of streamlines computed for any porosity, $0 < \epsilon < 1$.

The microscopic disturbance, near the reference sphere, of the macroscopic mainstream $\underline{\hat{U}} = [\hat{U}, 0, 0]$ is effectively confined to a spherical region concentric with the unit cell, having a radius twice that of the unit cell.*

## EXPERIMENTAL WORK

Although a considerable amount of experimental data for permeabilities of homogeneous and isotropic swarms of spherical particles was available, a particularly carefully arranged and executed series of measurements was undertaken (see Figure 7). Particular care was taken with respect to shape and size of the spherical particles (precision ground stainless steel balls of 1-mm diameter). The spheres were randomly packed into a heavy-wall brass cylinder of 50.8-mm inner diameter, submerged under distilled water to preclude the inclusion of air bubbles into the system. The porosity of the system was determined by a weighing technique; it was found to be 0.379. A steady stream of distilled water was passed through the system; its flow rate was set and controlled by means of a hydraulic flow controller. A pressure transducer, periodically calibrated against a high precision differential manometer, was used to measure the pressure difference. The permeability was then calculated using Equation (23).

Several measurements were then performed in the range of Reynolds number: $0.08 \leqq Re \leqq 1.40$ (the Reynolds number based upon the particle diameter (that is, 1 mm) and superficial (that is, assembly-averaged) flow velocity. The results are listed in Table 2 in normalized form. The point corresponding to $Re = 0.08$, which is within the creeping flow regime (that is, $Re < 0.1$ for spherical particles), is included also in Figures 4 and 5; it conforms closely to previously reported experimental data.

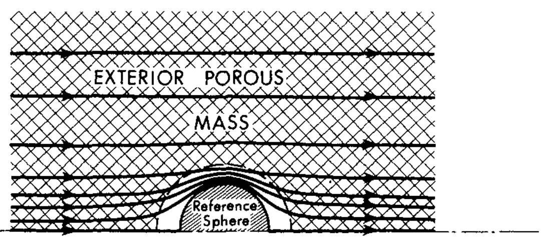
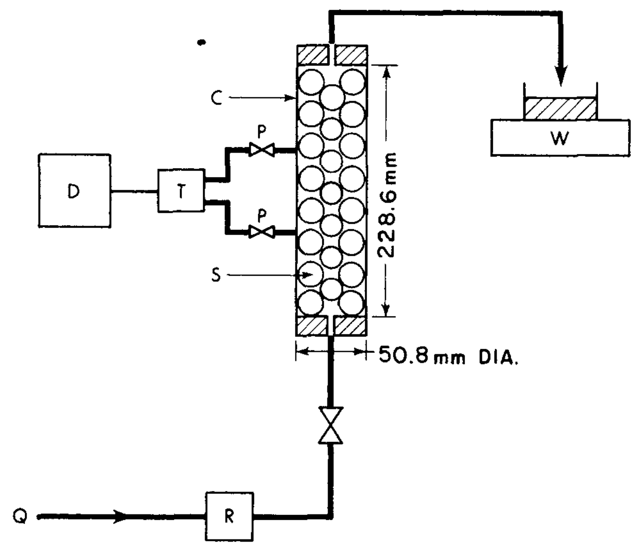



**Fig. 6. Typical streamlines for incompressible creeping flow through the modeled system (for the representative porosity $\epsilon = 0.7$).**

## DISCUSSION OF PREVIOUS WORK

Numerous experimental and theoretical studies concerning incompressible fluid flow relative to a swarm of spherical particles have been reported in the literature. These have been reviewed in conscientious detail by Scheidegger (1960) and by Happel and Brenner (1965); hence, the discussion can here be limited to those theoretical investigations which are considered of particular interest in the present context.

Bruggemann (1935) suggested a geometric model for his theoretical studies of electric conduction within a swarm of spherical particles. It consisted of a typical particle embedded immediately (that is, without a concentric spherical shell of associated pore space) within the exterior




KEY:

C CYLINDRICAL BRASS CELL

S STAINLESS STEEL SPHERES (1mm dia. bearing balls)

Q DISTILLED WATER SUPPLY

P PRESSURE TAPS (spaced 76.2mm apart)

W WEIGHING SCALE

R CONSTANT FLOW REGULATOR (Moore Instrument Co. model 63BU)

T PRESSURE TRANSDUCER (Whittaker Corp. model KP15)

D DIGITAL VOLTMETER (Non-Linear Systems Inc. model X-3)

**Fig. 7. Equipment used for the experimental determination of the permeability of a pack of spheres.**

* In the application of the geometric model to the theoretical study of diffusive flow processes occurring within an homogeneous and isotropic swarm of spherical particles (Neale and Nader, 1973), the microscopic disturbance of the macroscopic mainstream was found to be totally restricted to within the unit cell.

Table 2. Experimental Permeability Data for an Homogeneous Swarm of Monosized Spheres

$\epsilon = 0.379$

| $Re$ | $\alpha$ | $W$ |
|---|---|---|
| 0.081 | 19.08 | 0.00768 |
| 0.093 | 19.12 | 0.00765 |
| 0.168 | 19.10 | 0.00766 |
| 0.192 | 19.09 | 0.00769 |
| 0.251 | 19.12 | 0.00765 |
| 0.290 | 19.10 | 0.00766 |
| 0.552 | 19.10 | 0.00766 |
| 0.670 | 19.13 | 0.00764 |
| 0.807 | 19.11 | 0.00765 |
| 0.932 | 19.14 | 0.00763 |
| 1.38 | 19.16 | 0.00762 |

porous mass. Brinkman (1947) adopted this geometric model for his theoretical studies of creeping fluid flow relative to a swarm of spherical particles. With this model he obtained the following prediction:

$$W(\epsilon) = \frac{1}{1 + \alpha + \alpha^2/3} = 1 - \frac{3(1-\epsilon)}{4}\left(\left(\frac{5+3\epsilon}{1-\epsilon}\right)^{1/2} - 1\right) \quad (33)$$

This prediction is unsatisfactory for $\epsilon < 0.5$; indeed, it predicts $W(1/3) = 0$ (compare Figure 5).

It is quite elucidating to compare the model of this investigation with the Brinkman model. When considering for the former $\beta \rightarrow \alpha$, we obtain in the limit $\beta = \alpha$ (zero thickness of the spherical shell of associated pore space) the Brinkman model. For the model of this investigation, this implies $\epsilon = 0$, by (1). Disregarding for the present that the prediction (31) with (16) is a function of the porosity, we consider

$$\lim_{\beta\rightarrow\alpha} W = \lim_{\beta\rightarrow\alpha} \frac{1}{\xi(\alpha, \beta)} = \frac{1}{1 + \alpha + \alpha^2/3} \quad (34)$$

This limit is identical with the Brinkman prediction (33). This comparison of the two models provides the explanation for the anomalous behavior of the Brinkman solution: it is a consequence of the fact that the Brinkman model simplifies too drastically; it disturbs in the modeling process the uniform porosity of the porous medium where such a disturbance is most damaging, that is, next to the reference sphere.

Happel (1958) proposed a remarkable cell-type model which is generally referred to as the free surface model; he postulated that each particle within a sedimenting cloud of spherical particles would effectively be limited to motion within a concentric volume of fluid; on the outer envelope of this volume, both the normal velocity component and the tangential shear stress are presumed to vanish. Using this representation, Happel derived the following prediction for swarms of monosized spheres:

$$W(\epsilon) = \frac{6 - 9(1-\epsilon)^{1/3} + 9(1-\epsilon)^{5/3} - 6(1-\epsilon)^2}{6 + 4(1-\epsilon)^{5/3}} \quad (35)$$

which is valid for flow through fixed beds as well as for sedimentation. Equation (35) yields values which are in good agreement with experimental data, as shown in Figure 5.

Kuwabara (1959) modified the free surface model by replacing the zero shear assumption at the outer envelope by one of zero vorticity. However, his predictions are less satisfactory than those of Happel.

Inspired by Kozeny's classical bundle of capillaries model, Carman (1937) developed the following semiempirical correlation for the permeability of a bed of monosized spheres of radius $R$:

$$k = \frac{R^2 \epsilon^3}{9K_c(1-\epsilon)^2} \quad (36)$$

which, together with (17) and (31), yields

$$W(\epsilon) = \frac{\epsilon^3}{2K_c(1-\epsilon)} \quad (37)$$

The parameter $K_c$ denotes the so-called "Carman constant" which must be determined by experiment. The commonly accepted value of $K_c$ for spherical particles is 5.0; with this value, Equation (37) becomes reasonably representative of experimental data in the porosity range $0.26 < \epsilon < 0.7$. This important correlation is shown in Figure 5.

A number of statistical models have been advanced to study fluid flow through porous media in general, some of which have been discussed by Scheidegger (1960). However, although such approaches are extremely interesting, they tend (at present) to be too qualitative in nature to permit any direct engineering application.

## APPLICABILITY OF THE PRESENT THEORY TO HINDERED SETTLING

Although the preceding analysis has been concerned with the study of flow through a fixed bed of spherical particles, the predictions obtained are also valid for describing the hindered settling of a fixed suspension of spheres (as well as the fluidization of a fixed swarm) since each process involves the relative motion of a continuous fluid phase and a fixed particulate phase. In the case of hindered settling, the permeability of the swarm $k$ has little physical meaning, but the quantity $W$ appearing in (31) can be shown to be equal to $U/U_0$ where $U$ here denotes the settling velocity of the suspension, and $U_0$ the settling velocity of one of the individual spheres in an infinite medium. The results displayed in Figure 5 are therefore directly applicable to the hindered settling of a fixed swarm of monosized spheres.

The presented theory is strictly valid for hindered settling (and fluidization) only when the component spheres within the suspension are fixed, that is, when they maintain their relative positions and orientations with respect to one another. This condition will seldom be encountered in practice since the particles will tend to rotate and translate within the swarm as a result of hydrodynamic interactions between the particles. A number of valuable theoretical contributions concerning the hindered settling of free suspensions, in which these interaction effects have been explicitly accounted for, have recently appeared in the literature [refer to Wacholder (1973) and Saffman (1973)]. Unfortunately, in practice, there is often an additional complicating effect, which was not considered in the preceding theories, but which was alluded to by Saffman. This concerns the inevitable electric charge which a solid surface acquires when it is immersed in an aqueous electrolyte. Since electrical neutrality must be maintained in such a system a diffuse layer of oppositely charged ions forms in the electrolyte near the solid surface, giving rise to what is commonly called an *electric double layer*. The presence of electric double layers in sedimentation experiments tends to reduce the settling velocity. The effects of these double layers can usually be neglected when using large particles (that is, $R >> 10\mu$) and concentrated electrolytes, but they are frequently too significant to be neglected when using small particles ($R \lesssim 10\mu$), especially at high particle concentrations. A recent contribution by Levine and Neale (1974) demonstrates how the previously discussed cell-models of Happel and Kuwabara can be adapted to predict the effects of electric double layers in situations involving aqueous electrolyte flow relative to swarms of spherical particles.

## SUMMARY: FURTHER APPLICATIONS OF THE PROPOSED GEOMETRIC MODEL

The presented results demonstrate that the proposed geometric model permits good physical representation of, and provides valuable insight into, the hydrodynamic flow process relative to an homogeneous and isotropic porous medium composed of impermeable spherical particles of radius $R$.

This result, together with the encouraging results reported in an earlier paper (Neale and Nader, 1973) for diffusive transport processes occurring within an homogeneous and isotropic porous medium, qualifies this geometric model for theoretical studies of compound transport processes involving two or more fundamental transport processes of either category.

In view of the encouraging results obtained, the geometric model is being employed in further theoretical studies of the physical properties of an homogeneous and isotropic swarm of spherical particles.

## ACKNOWLEDGMENT

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## NOTATION

$F$ = magnitude of hydrodynamic resistance force
$k$ = permeability of porous medium
$N$ = number of spheres
$p$ = pressure (referred to a datum plane)
$R$ = radius of sphere
$S$ = radius of unit cell
$u$ = velocity of flowing fluid
$U$ = mainstream fluid velocity
$W$ = function defined in (31)
$(x, y, z)$ = Cartesian coordinates
$(r, \Theta, \varphi)$ = spherical coordinates

### Greek Letters

$\alpha$ = normalized radius of sphere $(R/\sqrt{k})$
$\beta$ = normalized radius of unit cell $(S/\sqrt{k})$
$\epsilon$ = porosity of porous medium
$\mu$ = viscosity of fluid
$\psi$ = stream-function
$\tau_{r\Theta}$ = tangential shear stress
$\xi$ = influence function defined in (15)
$\chi$ = normalized radial coordinate $(r/\sqrt{k})$

### Subscripts

$r$ = denotes radial direction
$\theta$ = denotes polar direction
— = designates vector quantities

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